Introduction to Differential Equations

What is a D.E.?

Is a statement of equality that relates an independent variable (t) a function $y = f(t)$ and one or more derivatives y', y'', $y^{(3)}$, etc or their differentials. Examples:

a) $y' = 2t$
b) $3t^2y'' + 11ty' - 3y = 0$
c) $(1 + t\mathbf{e}^{ty})dy = -(1 + y\mathbf{e}^{ty})dt$

What is a solution? How do we verify?

A solution to a D.E. on some interval is a function (or functions) $y = f(t)$ such that y and its derivatives produce an identity in the original equation.

Ex.1 Consider the equation $3t^2y'' = 3y - 11ty'$.

a) Is $f(t) = t^2$ a solution?

$f(t) = y(t) = t^2$
$y' = 2t$
$y'' = 2$
} Plug into D.E.

$3t^2(2) = 3(t) - 11t(2t)$
$6t^2 = 3t - 22t$
$y(t) = t^2$ is Not a solution

b) Is $f(t) = t^{-3}$ a solution?

$f(t) = y(t) = t^{-3}$
$y' = -3t^{-4}$
$y'' = 12t^{-5}$
} Plug into DE

$3t^2(12t^{-5}) = 3t^{-3} - 11t(-3t^{-4})$
$6t^{-3} = 3t^{-3} + 33t^{-3}$

Yes, is a solution

Ex.2 Verify that $y(t) = A + B\mathbf{e}^{-9t}$ is a solution to equation $y'' + 9y' = 0$ on the interval $(-\infty, \infty)$.

$y'(t) = -9Be^{-9t}$
$y''(t) = 81Be^{-9t}$ } plug into DE

$-9Be^{-9t} + 81Be^{-9t} = 0$ ✓
$y'' + 9y' = 0$

the family of solutions $y(t) = A + Be^{-9t}$ is a solution

How do we classify D.E.'s?

I. By order: The order of a D.E. is the order of the highest derivative that appear in the differential equation. Examples:

a) $y' = 2t$ is a first order differential equation.

b) $\mathbf{e}^t y^{(3)} + 5\left(y^{(2)}\right)^4 - t^2 y' = 0$ is a third order differential equation.

II. Linear vs nonlinear: A linear D.E. has the form

$$g_n(t)y^{(n)} + g_{n-1}(t)y^{(n-1)} + \cdots g_2(t)y^{(2)} + g_1(t)y' + g_0(t)y = h(t)$$

Examples:

not $\emptyset$ - non-homogeneous

a) $t^5 y^{(3)} + \sin(t)y' = \cos(t^2)$ is a third order **linear** differential equation, **why?**

3rd order derivative

follow the above form

b) $\cos(t)y'' + tyy' = t^3$ is a second order **nonlinear** differential equation, **why?**

second order derivative

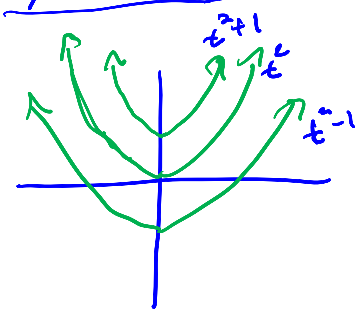
does not follow above form

Remark: In the linear case, if $h(t)$ is the zero function we say that the equation is linear and homogeneous. Otherwise, we say that the equation is linear and nonhomogeneous.

Ex.3 Given the equation $y' = 2t$.

a) What is the general solution? Describe the general solution geometrically.

$y(t) = t^2 + 2$ – family of solutions



$y(0) = -3$ – initial condition

Initial Value Problem (IVP) – only one solution

b) What if we were given: $y' = 2t,\ y(0) = -3$? What is the solution to this problem?

$y = t^2 + 2 + C$

$y = t^2 - 3$ ← guess not in lecture

Initial Value Problem (I.V.P.) This is a differential equation with an initial condition.